DE GRUYTER Adv. Nonlinear Anal. 2017; 6(4): 407–426

Research Article

Eduardo Colorado*

On the existence of bound and ground states for some coupled nonlinear Schrödinger–Korteweg–de Vries equations

DOI: 10.1515/anona-2015-0181
Received December 30, 2015; accepted February 8, 2016

Abstract: We show the existence of positive bound and ground states for a system of coupled nonlinear Schrödinger–Korteweg–de Vries equations. More precisely, we prove that there exists a positive radially symmetric ground state if either the coupling coefficient satisfies $\beta > \Lambda$ (for an appropriate constant $\Lambda > 0$) or if $\beta > 0$ under appropriate conditions on the other parameters of the problem. We also prove that there exists a positive radially symmetric bound state if either $0 < \beta$ is sufficiently small or if $0 < \beta < \Lambda$ under some appropriate conditions on the parameters. These results give a classification of positive solutions as well as the multiplicity of positive solutions. Furthermore, we study systems with more general power nonlinearities and systems with more than two nonlinear Schrödinger–Korteweg–de Vries equations. Our variational approach (working on the full energy functional without the L^2-mass constraint) improves many previously known results and also allows us to show new results for some range of parameters not considered in the past.

Keywords: Variational methods, ground states, bound states, critical point theory, perturbation methods

MSC 2010: 34B18, 34G20, 35J50, 35Q53, 35Q55

1 Introduction

The aim of this work is to study the system of coupled nonlinear Schrödinger–Korteweg–de Vries equations (NLS–KdV for short)

$$\begin{cases} if_t + f_{xx} + |f|^2 f + \beta fg = 0, \\ g_t + g_{xxx} + gg_x + \dfrac{1}{2}\beta(|f|^2)_x = 0, \end{cases} \tag{1.1}$$

where $f = f(x,t) \in \mathbb{C}$, while $g = g(x,t) \in \mathbb{R}$ and $\beta \in \mathbb{R}$ is the real coupling coefficient. System (1.1) appears in phenomena of interactions between short and long dispersive waves arising in fluid mechanics, such as the interactions of capillary-gravity water waves. Indeed, f represents the short wave, while g stands for the long wave; see [2, 18, 23] and the references therein for more details.

We look for solitary "traveling-wave" solutions, namely, solutions to (1.1) of the form

$$(f(x,t), g(x,t)) = (e^{i\omega t} e^{i\frac{c}{2}x} u(x-ct), v(x-ct)) \quad \text{with } u, v \text{ real functions.} \tag{1.2}$$

Choosing $\lambda_1 = \omega + \frac{c^2}{4}$, $\lambda_2 = c$, we get that u, v solve the problem

$$\begin{cases} -u'' + \lambda_1 u = u^3 + \beta uv, \\ -v'' + \lambda_2 v = \dfrac{1}{2}v^2 + \dfrac{1}{2}\beta u^2. \end{cases} \tag{1.3}$$

***Corresponding author: Eduardo Colorado:** Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid; and Instituto de Ciencias Matemáticas, ICMAT (CSIC-UAM-UC3M-UCM), C/ Nicolás Cabrera 13–15, 28049 Madrid, Spain, e-mail: eduardo.colorado@uc3m.es

This system has been previously studied by Dias, Figueira and Oliveira in [19]. Also, a generalization of (1.3), see (5.2), has been previously analyzed by the same authors in [20] and by Albert and Bhattarai in [3]. A comparison with our results will be done at the end of this introduction.

The main goal of this work is threefold. First, we give a classification of positive solutions of (1.3) proving the existence of positive even ground states under the hypotheses that
(i) the coupling coefficient satisfies $\beta > \Lambda > 0$ for an appropriate constant Λ, see Theorem 4.1;
(ii) $\beta > 0$ and $\lambda_2 \gg 1$, see Theorem 4.3.
Moreover, we prove the existence of positive even bound states when
(i) $0 < \beta \ll 1$, see Theorem 4.4, where we also give a bifurcation result;
(ii) $0 < \beta < \Lambda$ and $\lambda_2 \gg 1$, see Theorem 4.3.
The coexistence of positive bound and ground states for $0 < \beta < \Lambda$ and λ_2 large is a great novelty and also a difference compared to the more studied systems of NLS equations in the last several years, see Remark 4.6 (ii). Second, we study a more general system than (1.3) with more general power nonlinearities, given by (5.2), for which we show that previous results for (1.3) hold with similar conditions on the coefficients. We also analyze a particular case of λ_1, λ_2 in which there exists an explicit positive solution. Third, we consider natural extensions of (1.3) to systems with more than two equations and we also deal with extensions to the cases $n = 2, 3$ for which although (1.1) has no sense, the stationary system (1.3) makes sense and can be seen, for example, as the stationary system when one looks for standing-wave solutions of the corresponding evolutionary system of NLS equations. For some of these extended problems, we show similar results as described above on the existence of positive radially symmetric bound and ground state solutions. Other systems with at least two NLS equations and at least one KdV equation will be analyzed (in more detail) in a forthcoming paper.

Except for the above results, it is relevant to point out that it is the first time that our variational procedure (in part developed in [5, 6] for systems of coupled NLS equations) is employed to study coupled NLS–KdV equations in an appropriate way, see Remark 6.2 (ii). Furthermore, as we will see, it seems to be better in many ways than the classical approach used before in the study of NLS–KdV systems. Note that although our method is not completely new since it uses some known variational techniques, it could be exploited to study related problems which researchers working with this kind of dispersive systems have not used in the past.

It is worth pointing out that, for any $\beta \in \mathbb{R}$, system (1.3) has a unique *semitrivial* positive solution $\boldsymbol{v}_2 = (0, V_2)$, where

$$V_2(x) = 3\lambda_2 \operatorname{sech}^2\Big(\frac{\sqrt{\lambda_2}}{2}x\Big)$$

is the unique positive even solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2$ in $W^{1,2}(\mathbb{R})$, see [28]. As might be expected, we look for different solutions from the one above. We are not interested in nonnegative solutions but in positive ones, and therefore different from $\boldsymbol{v}_2$.

As we mentioned above, a comparison with our results and the previous works [3, 19, 20] is in order. In [19], Dias, Figueira and Oliveira studied (1.3) in the particular case $\lambda_1 = \lambda_2$ and they proved the existence of nonnegative bound state solutions when the coupling parameter satisfies $\beta > \frac{1}{2}$. In this paper, we improve that result in three ways. First, we consider λ_1 not necessarily equal to λ_2 and we prove not only the existence of nonnegative bound states but also of positive even ground states for β greater than a constant $\Lambda > 0$, defined by (3.8), for which, in the setting of [19], we have $\Lambda \le \frac{1}{2}$. Second, we show the existence of positive even bound states when $0 < \beta \ll 1$, a case not studied in [19]. Third, we also show that if λ_2 is sufficiently large, then there exists a positive even ground state for every $\beta > 0$ (with λ_1 not necessarily equal to λ_2) and a positive even bound state provided $0 < \beta < \Lambda$. In [20], among other results, Dias, Figueira and Oliveira studied system (5.2) with $2 < q < 5$, $p \in \{2, 3, 4\}$, $\mu_2 = p + 1$ and they established ([20, Theorem 4.1]) the existence of a nontrivial bound state solution for all $\beta > 0$ if $p = 3, 4$ and $\beta > 3$ if $p = 2$. Finally, in [3], Albert and Bhattarai studied, among other topics, system (5.2) in a more general setting than in [20]. More precisely, they considered the case $2 \le q < 5$, $2 \le p < 5$ with p a rational number with odd denominator and they proved the existence of a positive even bound state for each $\beta > 0$, improving the above cited result in [20]. In this paper, we consider the case $2 \le p < \infty$, $2 \le q < \infty$ and we prove that there exists a positive even ground state of (5.2) if either $\beta > \Lambda$ or $\beta > 0$, $q > 2p - 2$ and for λ_2 large enough. Concerning bound states, we show the existence of a positive even bound state of (5.2) if either $0 < \beta < \Lambda$, $q > 2p - 2$ and for λ_2 large enough or $0 < \beta \ll 1$, proving also in this last case a bifurcation result for the bound state we find.

By the discussion of problem (5.2) above, we improve and extend some of the results in [3, 20]. Additionally, we establish some new results for (5.2) like the multiplicity one of coexistence of positive bound and ground states in some range of the parameters. Finally, we analyze extended systems of (1.3) with more than two equations, which, to our knowledge, have not been considered previously in the literature. Also, we study the qualitative and quantitative properties of the explicit solutions of (1.3). We note that a preliminary announcement of some results in the present work appeared in [17].

The paper is organized as follows. In Section 2, we introduce the functional framework, notation and we give some definitions. Next, we define the Nehari manifold in Section 3, proving some of its properties, we establish a useful measure theoretic lemma and we prove a result dealing with the qualitative properties of the semitrivial solution. Section 4 is divided into two subsections, the first one establishing the existence of ground states and the second one dealing with the existence of bound states. In Section 5, we study a system with more general power nonlinearities, proving similar results as in the previous case with the appropriate changes. Section 6 contains two subsections, where in the first one we deal with an explicit solution, while the second one is devoted to the study of natural extensions to systems with more than two equations.

2 Functional setting and notation

Let E denote the Sobolev space $W^{1,2}(\mathbb{R})$ that can be defined as the completion of $\mathcal{C}_0^1(\mathbb{R})$ endowed with the norm

$$\|u\| = \sqrt{(u|u)}$$

which comes from the scalar product

$$(u|w) = \int_{\mathbb{R}} (u'w' + uw)\, dx.$$

We will denote (taking into account that $\lambda_1, \lambda_2 > 0$) the equivalent norms and scalar products in E as

$$\|u\|_j = \|u\|_{\lambda_j} = \left(\int_{\mathbb{R}} (|u'|^2 + \lambda_j u^2)\, dx \right)^{\frac{1}{2}},$$

$$(u|v)_j = (u|v)_{\lambda_j} = \int_{\mathbb{R}} (u' \cdot v' + \lambda_j uv)\, dx, \qquad j = 1, 2.$$

Let us define the product Sobolev space $\mathbb{E} = E \times E$. The elements in $\mathbb{E}$ will be denoted by $\boldsymbol{u} = (u, v)$ and $\boldsymbol{0} = (0, 0)$. We will take

$$\|\boldsymbol{u}\| = \sqrt{\|u\|_1^2 + \|v\|_2^2}$$

as a norm in $\mathbb{E}$.

For $\boldsymbol{u} = (u, v) \in \mathbb{E}$, the notation $\boldsymbol{u} \geq \boldsymbol{0}$, resp. $\boldsymbol{u} > \boldsymbol{0}$, means that $u, v \geq 0$, resp. $u, v > 0$. We denote by H the space of even (radially symmetric) functions in E and we denote $\mathbb{H} = H \times H$.

We define the functionals

$$I_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{4}\int_{\mathbb{R}} u^4 dx, \quad I_2(v) = \frac{1}{2}\|v\|_2^2 - \frac{1}{6}\int_{\mathbb{R}} v^3\, dx, \quad u, v \in E,$$

and

$$\Phi(\boldsymbol{u}) = I_1(u) + I_2(v) - \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v\, dx, \quad \boldsymbol{u} \in \mathbb{E}.$$

We also write

$$G_\beta(\boldsymbol{u}) = \frac{1}{4}\int_{\mathbb{R}} u^4\, dx + \frac{1}{6}\int_{\mathbb{R}} v^3\, dx + \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v\, dx, \quad \boldsymbol{u} \in \mathbb{E},$$

and, using this notation, we can rewrite the energy functional as

$$\Phi(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2 - G_\beta(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{E}.$$

Definition 2.1. We say that $\boldsymbol{u} \in \mathbb{E}$ is a nontrivial *bound state* of (1.3) if $\boldsymbol{u}$ is a nontrivial critical point of Φ. A bound state $\tilde{\boldsymbol{u}}$ is called a *ground state* if its energy is minimal among all the nontrivial bound states, namely,

$$\Phi(\tilde{\boldsymbol{u}}) = \min\{\Phi(\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{E} \setminus \{\boldsymbol{0}\}, \Phi'(\boldsymbol{u}) = 0\}. \tag{2.1}$$

3 Nehari manifold and key results

We will work mainly in $\mathbb{H}$. Setting

$$\Psi(\boldsymbol{u}) = (\nabla\Phi(\boldsymbol{u})|\boldsymbol{u}) = (I_1'(u)|u) + (I_2'(v)|v) - \frac{3}{2}\beta\int_{\mathbb{R}} u^2 v\,dx,$$

we define the corresponding Nehari manifold

$$\mathcal{N} = \{\boldsymbol{u} \in \mathbb{H} \setminus \{\boldsymbol{0}\} : \Psi(\boldsymbol{u}) = 0\}.$$

Then, one has that

$$(\nabla\Psi(\boldsymbol{u})|\boldsymbol{u}) = -\|\boldsymbol{u}\|^2 - \int_{\mathbb{R}} u^4\,dx < 0 \quad \text{for all } \boldsymbol{u} \in \mathcal{N}, \tag{3.1}$$

thus, $\mathcal{N}$ is a smooth manifold locally near any point $\boldsymbol{u} \neq \boldsymbol{0}$ with $\Psi(\boldsymbol{u}) = 0$. Moreover, $\Phi''(\boldsymbol{0}) = I_1''(0) + I_2''(0)$ is positive definite, so we infer that $\boldsymbol{0}$ is a strict minimum for Φ. As a consequence, $\boldsymbol{0}$ is an isolated point of the set $\{\Psi(\boldsymbol{u}) = 0\}$, proving, on one hand, that $\mathcal{N}$ is a smooth complete manifold of codimension 1 and, on the other hand, that there exists a constant $\rho > 0$ such that

$$\|\boldsymbol{u}\|^2 > \rho \quad \text{for all } \boldsymbol{u} \in \mathcal{N}. \tag{3.2}$$

Furthermore, (3.1) and (3.2) obviously imply that $\boldsymbol{u} \in \mathbb{H} \setminus \{\boldsymbol{0}\}$ is a critical point of Φ if and only if $\boldsymbol{u} \in \mathcal{N}$ is a critical point of Φ constrained on $\mathcal{N}$.

Remark 3.1. (i) By the previous arguments, the Nehari manifold $\mathcal{N}$ is a natural constraint of Φ. Also, it is relevant to point out that when working on the Nehari manifold, the functional Φ takes the form

$$\Phi|_{\mathcal{N}}(\boldsymbol{u}) = \frac{1}{6}\|\boldsymbol{u}\|^2 + \frac{1}{12}\int_{\mathbb{R}} u^4\,dx =: F(\boldsymbol{u}) \tag{3.3}$$

and, by plugging (3.2) into (3.3), we have

$$\Phi(\boldsymbol{u}) \geq \frac{1}{6}\|\boldsymbol{u}\|^2 > \frac{1}{6}\rho \quad \text{for all } \boldsymbol{u} \in \mathcal{N}. \tag{3.4}$$

Therefore, (3.4) shows that the functional Φ is bounded from below on $\mathcal{N}$, so one can try to minimize it on the Nehari manifold.

(ii) With respect to the Palais–Smale condition, we recall that in the one-dimensional case one cannot expect a compact embedding of E into $L^q(\mathbb{R})$ for $2 < q < \infty$. Indeed, working on H (the radial or even case) is not true as well, see [30, Remarque I.1]. However, we will show that, for a Palais–Smale sequence, we can find a subsequence for which the weak limit is a solution. This fact along with some properties of the Schwarz symmetrization will allow us to prove the existence of positive even ground states in Theorem 4.1. With some extra work, one could also consider nonnegative radially decreasing functions, where the required compactness follows from Berestycki and Lions [11].

Due to the lack of compactness mentioned in Remark 3.1 (ii) above, we state a measure theoretic result given in [31] that we will use in the proof of Theorem 4.1.

Lemma 3.2. *If $2 < q < \infty$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} |u|^q\,dx \le C\Big(\sup_{z\in\mathbb{R}} \int_{|x-z|<1} |u(x)|^2\,dx\Big)^{\frac{q-2}{2}} \|u\|_E^2 \quad \text{for all } u \in E. \tag{3.5}$$

See [15] for an extension of this lemma to fractional Sobolev spaces and for an application to a fractional system of NLS equations.

Let V denote the unique positive even solution of $-v'' + v = v^2$, $v \in H$, see [28]. Setting

$$V_2(x) = 2\lambda_2 V\big(\sqrt{\lambda_2}x\big) = \frac{3\lambda_2}{\cosh^2\big(\frac{\sqrt{\lambda_2}}{2}x\big)}, \tag{3.6}$$

one has that V_2 is the unique positive solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2$ in H. Hence, $\boldsymbol{v}_2 := (0, V_2)$ is a particular solution of (1.3) for any $\beta \in \mathbb{R}$ and, moreover, it is the unique nonnegative semitrivial solution of (1.3). We also define the corresponding Nehari manifold

$$\mathcal{N}_2 = \{v \in H : (I_2'(v)|v) = 0\} = \Big\{v \in H : \|v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}} v^3\,dx = 0\Big\}.$$

Let us denote by $T_{\boldsymbol{v}_2}\mathcal{N}$ the tangent space of $\mathcal{N}$ at $\boldsymbol{v}_2$. Since

$$\boldsymbol{h} = (h_1, h_2) \in T_{\boldsymbol{v}_2}\mathcal{N} \quad \text{if and only if} \quad (V_2|h_2)_2 = \frac{3}{4}\int_{\mathbb{R}} V_2^2 h_2\,dx,$$

it follows that

$$(h_1, h_2) \in T_{\boldsymbol{v}_2}\mathcal{N} \quad \text{if and only if} \quad h_2 \in T_{V_2}\mathcal{N}_2. \tag{3.7}$$

Proposition 3.3. *There exists $\Lambda > 0$ such that*
(i) *if $\beta < \Lambda$, then $\boldsymbol{v}_2$ is a strict local minimum of Φ constrained on $\mathcal{N}$;*
(ii) *for any $\beta > \Lambda$, $\boldsymbol{v}_2$ is a saddle point of Φ constrained on $\mathcal{N}$. Moreover, $\inf_{\mathcal{N}} \Phi < \Phi(\boldsymbol{v}_2)$.*

Proof. For (i), we define

$$\Lambda = \inf_{\varphi\in H\setminus\{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}} V_2\varphi^2\,dx}. \tag{3.8}$$

For $\boldsymbol{h} \in T_{\boldsymbol{v}_2}\mathcal{N}$, one has

$$\Phi''(\boldsymbol{v}_2)[\boldsymbol{h}]^2 = \|h_1\|_1^2 + I_2''(V_2)[h_2]^2 - \beta\int_{\mathbb{R}} V_2 h_1^2\,dx. \tag{3.9}$$

Let us take $\boldsymbol{h} = (h_1, h_2) \in T_{\boldsymbol{v}_2}\mathcal{N}$. By (3.7), $h_2 \in T_{V_2}\mathcal{N}_2$ and using that V_2 is the minimum of I_2 on $\mathcal{N}_2$, there exists a constant $c > 0$ such that

$$I_2''(V_2)[h_2]^2 \ge c\|h_2\|_2^2. \tag{3.10}$$

Using (3.10) together with (3.9), for $\beta < \Lambda$, there exists another constant $c_1 > 0$ such that

$$\Phi''(\boldsymbol{v}_2)[\boldsymbol{h}]^2 \ge c_1(\|h_1\|_1^2 + \|h_2\|^2). \tag{3.11}$$

Notice that $\Phi'(\boldsymbol{v}_2) = 0$ implies that $D^2\Phi_{\mathcal{N}}(\boldsymbol{v}_2)[\boldsymbol{h}]^2 = \Phi''(\boldsymbol{v}_2)[\boldsymbol{h}]^2$ for all $\boldsymbol{h} \in T_{\boldsymbol{v}_2}\mathcal{N}$ and, thus, using (3.11), we infer that $\boldsymbol{v}_2$ is a local strict minimum of Φ on $\mathcal{N}$.

For (ii), according to (3.7), $\boldsymbol{h} = (h_1, 0) \in T_{\boldsymbol{v}_2}\mathcal{N}$ for any $h_1 \in H$. We have that, for $\beta > \Lambda$, there exists $\widetilde{h} \in H$ with

$$\Lambda < \frac{\|\widetilde{h}\|_1^2}{\int_{\mathbb{R}} V_2 \widetilde{h}^2\, dx} < \beta,$$

thus, taking $\boldsymbol{h}_0 = (\widetilde{h}, 0) \in T_{\boldsymbol{v}_2}\mathcal{N}$, by (3.9) we get

$$\Phi''(\boldsymbol{v}_2)[\boldsymbol{h}_0]^2 = \|\widetilde{h}\|_1^2 - \beta \int_{\mathbb{R}} V_2 \widetilde{h}^2\, dx < 0 \quad \text{for all } \beta > \Lambda. \qquad \square$$

Remark 3.4. If one considers $\lambda_1 = \lambda_2$ as in [19], taking $\boldsymbol{h}_0 = (V_2, 0) \in T_{\boldsymbol{v}_2}\mathcal{N}$ in the proof of Proposition 3.3 (ii) yields

$$\Phi''(\boldsymbol{v}_2)[\boldsymbol{h}_0]^2 = \|V_2\|_2^2 - \beta \int_{\mathbb{R}} V_2^3\, dx = (1 - 2\beta)\|V_2\|_2^2 < 0 \quad \text{provided} \quad \beta > \frac{1}{2}.$$

See also Remark 4.2.

4 Main results

4.1 Existence of ground states

Concerning the existence of ground state solutions of (1.3), our first result is the following theorem.

Theorem 4.1. *Suppose that $\beta > \Lambda$. Then, system* (1.3) *has a positive even ground state $\widetilde{\boldsymbol{u}} = (\widetilde{u}, \widetilde{v})$.*

To prove this theorem we are going to work on the full Nehari manifold $\widetilde{\mathcal{N}} = \{\boldsymbol{u} \in \mathbb{E} \setminus \boldsymbol{0} : \Psi(\boldsymbol{u}) = 0\}$, which satisfies similar properties than $\mathcal{N}$.

Proof. We divide the proof into two steps. In the first step, we prove that $\inf_{\mathcal{N}} \Phi$ is achieved at some positive function $\widetilde{\boldsymbol{u}} \in \mathbb{E}$, while in the second step we show that $\widetilde{\boldsymbol{u}}$ is indeed even.

Step 1. By Ekeland's variational principle, see [21], there exists a Palais–Smale sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\boldsymbol{u}_k) \to c = \inf_{\mathcal{N}} \Phi, \quad \nabla_{\mathcal{N}}\Phi(\boldsymbol{u}_k) \to 0. \tag{4.1}$$

By (3.3) one easily finds that $\{\boldsymbol{u}_k\}$ is a bounded sequence on $\mathbb{E}$ and, after relabeling, we can assume that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ weakly in $\mathbb{E}$, $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $\mathbb{L}^q_{\text{loc}}(\mathbb{R}) = L^q_{\text{loc}}(\mathbb{R}) \times L^q_{\text{loc}}(\mathbb{R})$ for every $1 \le q < \infty$ and $\boldsymbol{u}_k \to \boldsymbol{u}$ a.e. Moreover, for the constrained gradient we have $\nabla_{\mathcal{N}}\Phi(\boldsymbol{u}_k) = \Phi'(\boldsymbol{u}_k) - \eta_k \Psi'(\boldsymbol{u}_k) \to 0$, where η_k is the corresponding Lagrange multiplier. Taking the scalar product with $\boldsymbol{u}_k$ and recalling that $(\Phi'(\boldsymbol{u}_k)|\boldsymbol{u}_k) = \Psi(\boldsymbol{u}_k) = 0$, we find that $\eta_k(\Psi'(\boldsymbol{u}_k)|\boldsymbol{u}_k) \to 0$ and this, together with (3.1)–(3.2), implies that $\eta_k \to 0$. Since, in addition, $\|\Psi'(\boldsymbol{u}_k)\| \le C < +\infty$, we deduce that $\Phi'(\boldsymbol{u}_k) \to 0$.

Let us define $\mu_k(x) = u_k^2(x) + v_k^2(x)$, where $\boldsymbol{u}_k = (u_k, v_k)$. We claim that there is no evanescence, i.e., there exist $R, C > 0$ such that

$$\sup_{z\in\mathbb{R}} \int_{|z-x|<R} \mu_k(x)\, dx \ge C > 0 \quad \text{for all } k \in \mathbb{N}. \tag{4.2}$$

On the contrary, if we suppose that

$$\sup_{z\in\mathbb{R}} \int_{|z-x|<R} \mu_k(x)\, dx \to 0,$$

by Lemma 3.2, applied in a similar way as in [15], we find that $\boldsymbol{u}_k \to \boldsymbol{0}$ strongly in $\mathbb{L}^q(\mathbb{R})$ for any $2 < q < \infty$ and, as a consequence, the weak limit $\boldsymbol{u}^* \equiv \boldsymbol{0}$. This is a contradiction since $\boldsymbol{u}_k \in \mathcal{N}$ and by (3.3), (3.4) and (4.1) we have

$$0 < \frac{1}{7}\rho < c + o_k(1) = \Phi(\boldsymbol{u}_k) = F(\boldsymbol{u}_k) \quad \text{with } o_k(1) \to 0 \text{ as } k \to \infty,$$

hence (4.2) is true and the claim is proved.

We observe that we can find a sequence of points $\{z_k\} \subset \mathbb{R}$ so that, by (4.2), the translated sequence $\overline{\mu}_k(x) = \mu_k(x + z_k)$ satisfies

$$\liminf_{k\to\infty} \int_{B_R(0)} \overline{\mu}_k \geq C > 0.$$

Taking into account that $\overline{\mu}_k \to \overline{\mu}$ strongly in $L^1_{\mathrm{loc}}(\mathbb{R})$, we obtain that $\overline{\mu} \not\equiv 0$. Therefore, defining $\overline{\boldsymbol{u}}_k(x) = \boldsymbol{u}_k(x+z_k)$, we have that $\overline{\boldsymbol{u}}_k$ is also a Palais–Smale sequence of Φ on $\mathcal{N}$, in particular, the weak limit of $\overline{\boldsymbol{u}}_k$, denoted by $\overline{\boldsymbol{u}}$, is a nontrivial critical point of Φ constrained on $\mathcal{N}$, so $\overline{\boldsymbol{u}} \in \mathcal{N}$. Thus, using (3.3) again, we have

$$\Phi(\overline{\boldsymbol{u}}) = F(\overline{\boldsymbol{u}}) \leq \liminf_{k\to\infty} F(\overline{\boldsymbol{u}}_k) = \liminf_{k\to\infty} \Phi(\overline{\boldsymbol{u}}_k) = c.$$

Furthermore, by Proposition 3.3 (ii), we know that necessarily $\Phi(\overline{\boldsymbol{u}}) < \Phi(\boldsymbol{v}_2)$.

Taking into account that $\overline{\boldsymbol{u}} \in \mathcal{N}$ and the maximum principle, we have $\overline{v} > 0$, thus, it is not difficult to show that $\tilde{\boldsymbol{u}} = |\overline{\boldsymbol{u}}| = (|\overline{u}|, |\overline{v}|) = (|\overline{u}|, \overline{v}) \in \mathcal{N}$ with

$$\Phi(\tilde{\boldsymbol{u}}) = \Phi(\overline{\boldsymbol{u}}) = \min\{\Phi(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{N}\}, \tag{4.3}$$

so we have $\tilde{\boldsymbol{u}} \geq \mathbf{0}$. Finally, by the maximum principle applied to the first equation and the fact that $\Phi(\tilde{\boldsymbol{u}}) < \Phi(\boldsymbol{v}_2)$, we get $\tilde{\boldsymbol{u}} > \mathbf{0}$.

Step 2. Setting $\boldsymbol{w} = |\tilde{\boldsymbol{u}}|$, there holds

$$\Phi(\boldsymbol{w}) = \Phi(\tilde{\boldsymbol{u}}), \quad \Psi(\boldsymbol{w}) = \Psi(\tilde{\boldsymbol{u}}). \tag{4.4}$$

For $\boldsymbol{w} = (w_1, w_2)$, we set $\boldsymbol{w}^\star = (w_1^\star, w_2^\star)$, where $w_j^\star$ is the Schwarz symmetric function associated to $w_j \geq 0$, $j = 1, 2$. Then, by the classical properties of the Schwarz symmetrization (see, for instance, [27]) we have

$$\|\boldsymbol{w}^\star\|^2 \leq \|\boldsymbol{w}\|^2, \quad G_\beta(\boldsymbol{w}^\star) \geq G_\beta(\boldsymbol{w}), \tag{4.5}$$

thus, in particular, $\Psi(\boldsymbol{w}^\star) \leq \Psi(\boldsymbol{w})$. Using the second identity of (4.4) and the fact that $\tilde{\boldsymbol{u}}$ is a critical point of Φ, we get $\Psi(\boldsymbol{w}) = \Psi(\tilde{\boldsymbol{u}}) = 0$. Furthermore, there exists a unique $t_0 > 0$ such that $t_0\boldsymbol{w}^\star \in \mathcal{N}$. In fact, t_0 comes from $\Psi(t_0\boldsymbol{w}^\star) = 0$, i.e.,

$$\|\boldsymbol{w}^\star\|^2 = t_0^2 \int_{\mathbb{R}} (w_1^\star)^4\,dx + t_0\left(\frac{1}{2}\int_{\mathbb{R}} (w_2^\star)^3\,dx + \frac{3}{2}\beta\int_{\mathbb{R}} (w_1^\star)^2 w_2^\star\,dx\right) \tag{4.6}$$

and using that $\Psi(\boldsymbol{w}) = 0$, (4.5)–(4.6) and the fact that $\boldsymbol{w} > \mathbf{0}$ and $t_0 > 0$, we get

$$\int_{\mathbb{R}} w_1^4\,dx + \frac{1}{2}\int_{\mathbb{R}} w_2^3\,dx + \frac{3}{2}\beta\int_{\mathbb{R}} w_1^2 w_2\,dx \geq t_0^2\int_{\mathbb{R}} w_1^4\,dx + t_0\left(\frac{1}{2}\int_{\mathbb{R}} w_2^3\,dx + \frac{3}{2}\beta\int_{\mathbb{R}} w_1^2 w_2\,dx\right).$$

Thus, clearly $t_0 \leq 1$ and, as a consequence,

$$\Phi(t_0\,\boldsymbol{w}^\star) = \frac{1}{6}t_0^2\|\boldsymbol{w}^\star\|^2 + \frac{1}{12}t_0^4\int_{\mathbb{R}} (w_1^\star)^4\,dx \leq \frac{1}{6}\|\boldsymbol{w}\|^2 + \frac{1}{12}\int_{\mathbb{R}} w_1^4\,dx = \Phi(\boldsymbol{w}). \tag{4.7}$$

Therefore, inequality (4.7) and the first identity of (4.4) yield

$$\Phi(t_0\boldsymbol{w}^\star) \leq \Phi(\boldsymbol{w}) < \Phi(\tilde{\boldsymbol{u}}) = \min\{\Phi(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{N}\},$$

with $t_0\,\boldsymbol{w}^\star \in \widetilde{\mathcal{N}}$ which proves that we have an even ground state. □

Remark 4.2. As we have mentioned at the introduction, see also Remark 3.4, in the setting of [19], i.e., $\lambda_1 = \lambda_2$ and $\beta > \frac{1}{2}$, we have found positive even ground state solutions in contrast with the nonnegative bound states found in [19].

The last result in this subsection deals with the existence of positive ground states of (1.3) not only for $\beta > \Lambda$, but also for $0 < \beta \leq \Lambda$, at least for λ_2 large enough.

Theorem 4.3. *There exists $\Lambda_2 > 0$ such that if $\lambda_2 > \Lambda_2$, then system* (1.3) *has an even ground state $\tilde{\boldsymbol{u}} > \mathbf{0}$ for every $\beta > 0$.*

Proof. Arguing in the same way as in the proof of Theorem 4.1, we initially have that there exists an even ground state $\tilde{\boldsymbol{u}} \geq \mathbf{0}$. Moreover, for $\beta > \Lambda$, in Theorem 4.1 we have proved that $\tilde{\boldsymbol{u}} > \mathbf{0}$. Now, we need to show that, for $\beta \leq \Lambda$, there holds indeed $\tilde{\boldsymbol{u}} > \mathbf{0}$, which follows from the maximum principle provided $\tilde{\boldsymbol{u}} \neq \boldsymbol{v}_2$. Taking into account Proposition (3.3) (i), $\boldsymbol{v}_2$ is a strict local minimum, but this does not allow us to prove that $\tilde{\boldsymbol{u}} \neq \boldsymbol{v}_2$. The new idea here consists of proving the existence of a function $\boldsymbol{u}_1 = (u_1, v_1) \in \mathcal{N}$ with $\Phi(\boldsymbol{u}_1) < \Phi(\boldsymbol{v}_2)$. To do so, since $\boldsymbol{v}_2 = (0, V_2)$ is a local minimum of Φ on $\mathcal{N}$ provided $0 < \beta < \Lambda$, we cannot find $\boldsymbol{u}_1$ in a neighborhood of $\boldsymbol{v}_2$ on $\mathcal{N}$. Thus, we define $\boldsymbol{u}_1 = t(V_2, V_2)$, where $t > 0$ is the unique value such that $\boldsymbol{u}_1 \in \mathcal{N}$.

Notice that $t > 0$ is given by $\Psi(\boldsymbol{u}_1) = 0$, i.e.,

$$\|(V_2, V_2)\|^2 = t^2 \int_{\mathbb{R}} V_2^4\,dx + \frac{1}{2} t(1 + 3\beta) \int_{\mathbb{R}} V_2^3\,dx. \tag{4.8}$$

Moreover,

$$\|(V_2, V_2)\|^2 = 2\|V_2\|_2^2 + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2\,dx = \int_{\mathbb{R}} V_2^3\,dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2\,dx, \tag{4.9}$$

hence, plugging (4.9) into (4.8), we get

$$t^2 \int_{\mathbb{R}} V_2^4\,dx + \frac{1}{2} t(1 + 3\beta) \int_{\mathbb{R}} V_2^3\,dx = \int_{\mathbb{R}} V_2^3\,dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2\,dx,$$

therefore, dividing the above expression by the L^1 norm of V_2^3, using that

$$\int_{\mathbb{R}} \cosh^{-8}(x)\,dx = \frac{32}{35}, \qquad \int_{\mathbb{R}} \cosh^{-6}(x)\,dx = \frac{16}{15}, \qquad \int_{\mathbb{R}} \cosh^{-4}(x)\,dx = \frac{4}{3}$$

and the definition of V_2 by (3.6), we find that

$$\frac{18}{7} \lambda_2 t^2 + \frac{1}{2} t(1 + 3\beta) - \left(1 + 5\frac{\lambda_1 - \lambda_2}{12\lambda_2}\right) = 0. \tag{4.10}$$

The energies of $\boldsymbol{u}_1, \boldsymbol{v}_2$ are given by

$$\Phi(t(V_2, V_2)) = \frac{1}{6} t^2 \left(\int_{\mathbb{R}} V_2^3\,dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2\,dx \right) + \frac{1}{12} t^4 \int_{\mathbb{R}} V_2^4\,dx, \quad \Phi(\boldsymbol{v}_2) = \frac{1}{12} \int_{\mathbb{R}} V_2^3\,dx.$$

Thus, for the unique $t > 0$ given by (4.9), we want to prove that we have

$$\frac{1}{6} t^2 \left(\int_{\mathbb{R}} V_2^3\,dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}} V_2^2\,dx \right) + \frac{1}{12} t^4 \int_{\mathbb{R}} V_2^4\,dx < \frac{1}{12} \int_{\mathbb{R}} V_2^3\,dx$$

and then, arguing as for (4.10), it is sufficient to prove that the inequality

$$\frac{18}{7} \lambda_2 t^4 + t^2 \left(2 + 5\frac{\lambda_1 - \lambda_2}{6\lambda_2}\right) - 1 < 0 \tag{4.11}$$

holds. Using (4.10) and the fact that $2 + 5\frac{\lambda_1 - \lambda_2}{6\lambda_2} > 0$ for every $\lambda_1, \lambda_2 > 0$ and further fixing $\beta > 0$ gives that (4.11) is satisfied provided λ_2 is sufficiently large, namely, $\lambda_2 > \Lambda_2 > 0$, proving that $\Phi(\boldsymbol{u}_1) < \Phi(\boldsymbol{v}_2)$, which concludes the proof. □

4.2 Existence of bound states

In this subsection, we establish the existence of bound states of (1.3). The first theorem deals with a perturbation framework in which we suppose that $\beta = \varepsilon\tilde{\beta}$ with $\tilde{\beta}$ fixed and independent of ε. Note that $\tilde{\beta}$ can be negative and $0 < \varepsilon \ll 1$. Then, we rewrite the energy functional Φ as Φ_ε to emphasize its dependence on ε, i.e.,

$$\Phi_\varepsilon(\boldsymbol{u}) = \Phi_0(\boldsymbol{u}) - \frac{1}{2}\varepsilon\widetilde{\beta}\int_{\mathbb{R}} u^2 v\,dx,$$

where $\Phi_0 = I_1 + I_2$.

Let us set $\boldsymbol{u}_0 = (U_1, V_2)$, where V_2 is given by (3.6) and U_1 is the unique positive solution of $-u'' + \lambda_1 u = u^3$ in H, see [14, 28]. The function U_1 has the explicit expression

$$U_1(x) = \frac{\sqrt{2\lambda_1}}{\cosh(\sqrt{\lambda_1}x)}. \tag{4.12}$$

Note also that U_1 satisfies the identity

$$\|U_1\|_1 = \inf_{u\in H\setminus\{0\}} \frac{\|u\|_1^2}{(\int_{\mathbb{R}} u^4\,dx)^{\frac{1}{2}}}. \tag{4.13}$$

Theorem 4.4. *There exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta}$, system (1.3) has an even bound state $\boldsymbol{u}_\varepsilon$ with $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$ as $\varepsilon \to 0$. Moreover, if $\beta > 0$, then $\boldsymbol{u}_\varepsilon > \boldsymbol{0}$.*

In order to prove this result, we follow some ideas of [16, Theorem 4.2] with appropriate modifications.

Proof of Theorem 4.4. It is well known that U_1 and V_2 are nondegenerate critical points of I_1 and I_2 on H, respectively, see [28]. Obviously, $\boldsymbol{u}_0$ is a nondegenerate critical point of Φ_0 acting on $\mathbb{H}$. Then, by the local inversion theorem, there exists a critical point $\boldsymbol{u}_\varepsilon$ of Φ_ε for any $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small, see [8] for more details. Moreover, $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$ on $\mathbb{H}$ as $\varepsilon \to 0$. To complete the proof, it remains to show that if $\beta > 0$, then $\boldsymbol{u}_\varepsilon > \boldsymbol{0}$.

Let us denote the positive part by $\boldsymbol{u}_\varepsilon^+ = (u_\varepsilon^+, v_\varepsilon^+)$ and the negative part by $\boldsymbol{u}_\varepsilon^- = (u_\varepsilon^-, v_\varepsilon^-)$. By (4.13) we have

$$\|u_\varepsilon^\pm\|_1^2 \ge \|U_1\|_1\bigg(\int_{\mathbb{R}}(u_\varepsilon^\pm)^4\,dx\bigg)^{\frac{1}{2}}. \tag{4.14}$$

Multiplying the second equation of (1.3) by v_ε^- and integrating over $\mathbb{R}$ we obtain

$$\|v_\varepsilon^-\|_2^2 = \int_{\mathbb{R}}(v_\varepsilon^-)^3\,dx + \varepsilon\widetilde{\beta}\int_{\mathbb{R}}(u_\varepsilon)^2 v_\varepsilon^-\,dx \le 0, \tag{4.15}$$

thus, $\|v_\varepsilon^-\|_2 = 0$, which implies that $v_\varepsilon = v_\varepsilon^+ \ge 0$. Furthermore, $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$ implies that $v_\varepsilon \to V_2$, which together with the maximum principle gives that $v_\varepsilon > 0$ provided ε is sufficiently small.

Multiplying now the first equation of (1.3) by $u_\varepsilon^\pm$ and integrating over $\mathbb{R}$ we obtain

$$\|u_\varepsilon^\pm\|_1^2 = \int_{\mathbb{R}}(u_\varepsilon^\pm)^4\,dx + \varepsilon\widetilde{\beta}\int_{\mathbb{R}}(u_\varepsilon^\pm)^2 v_\varepsilon\,dx \le \int_{\mathbb{R}}(u_\varepsilon^\pm)^4\,dx + \varepsilon\widetilde{\beta}\bigg(\int_{\mathbb{R}}(u_\varepsilon^\pm)^4\,dx\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}}v_\varepsilon^2\,dx\bigg)^{\frac{1}{2}}.$$

This, together with (4.14), yields

$$\|u_\varepsilon^\pm\|_1^2 \le \frac{\|u_\varepsilon^\pm\|_1^4}{\|U_1\|_1^2} + \varepsilon\theta_\varepsilon\frac{\|u_\varepsilon^\pm\|_1^2}{\|U_1\|_1}, \tag{4.16}$$

where

$$\theta_\varepsilon = \widetilde{\beta}\bigg(\int_{\mathbb{R}} v_\varepsilon^2\bigg)^{\frac{1}{2}}.$$

Hence, if $\|u_\varepsilon^\pm\| > 0$, one infers that

$$\|u_\varepsilon^\pm\|_1^2 \ge \|U_1\|_1^2 + o(1), \tag{4.17}$$

where $o(1) = o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Using again $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$, we have $u_\varepsilon \to U_1 > 0$ and, as a consequence, $\|u_\varepsilon^+\| > 0$ for ε small enough. Thus, (4.17) gives

$$\|\boldsymbol{u}_\varepsilon^+\|^2 = \|u_\varepsilon^+\|_1^2 + \|v_\varepsilon^+\|_2^2 \geq \|U_1\|_1^2 + o(1). \tag{4.18}$$

To reach a contradition, suppose now that $\|u_\varepsilon^-\|_1 > 0$. Then, as for (4.18), we obtain

$$\|\boldsymbol{u}_\varepsilon^-\|^2 = \|u_\varepsilon^-\|_1^2 + \|v_\varepsilon^-\|_2^2 \geq \|U_1\|_1^2 + o(1). \tag{4.19}$$

On one hand, using (4.18)–(4.19), we find that

$$\begin{aligned}\Phi(\boldsymbol{u}_\varepsilon) &= \frac{1}{6}\|\boldsymbol{u}_\varepsilon\|^2 + \frac{1}{12}\int_{\mathbb{R}} u_\varepsilon^4\,dx \\ &= \frac{1}{6}[\|\boldsymbol{u}_\varepsilon^+\|^2 + \|\boldsymbol{u}_\varepsilon^-\|^2] + \frac{1}{12}\int_{\mathbb{R}}[(u_\varepsilon^+)^4 + (u_\varepsilon^-)^4]\,dx \\ &\geq \frac{1}{6}\|\boldsymbol{u}_0\|^2 + \frac{1}{6}\|U_1\|_1^2 + \frac{1}{12}\int_{\mathbb{R}} U_1^4\,dx + o(1).\end{aligned} \tag{4.20}$$

On the other hand, since $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$, we have

$$\Phi(\boldsymbol{u}_\varepsilon) = \frac{1}{6}\|\boldsymbol{u}_\varepsilon\|^2 + \frac{1}{12}\int_{\mathbb{R}} u_\varepsilon^4\,dx \to \frac{1}{6}\|\boldsymbol{u}_0\|^2 + \frac{1}{12}\int_{\mathbb{R}} U_1^4\,dx, \tag{4.21}$$

which is in contradiction with (4.20), proving that $u_\varepsilon \geq 0$.

In conclusion, we have proved that $v_\varepsilon > 0$ and $u_\varepsilon \geq 0$. To prove the positivity of u_ε, using once more that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0$ and $\beta = \varepsilon\widetilde{\beta} \geq 0$, we can apply the maximum principle to the first equation of (1.3), which implies that $u_\varepsilon > 0$ and, finally, $\boldsymbol{u}_\varepsilon > \boldsymbol{0}$. □

From the existence of a positive ground state established in Theorem 4.1 for $\beta > \Lambda$ and, more precisely, in Theorem 4.3 for $\beta > 0$, provided λ_2 is sufficiently large, we can show the existence of a different positive bound state of (1.3) in the following theorem.

Theorem 4.5. *Under the hypotheses of Theorem 4.3 and* $0 < \beta < \Lambda$, *there exists an even bound state* $\boldsymbol{u}^* > \boldsymbol{0}$ *with* $\Phi(\boldsymbol{u}^*) > \Phi(\boldsymbol{v}_2)$.

Proof. The positive ground state $\widetilde{\boldsymbol{u}}$ found in Theorem 4.3 satisfies $\Phi(\widetilde{\boldsymbol{u}}) < \Phi(\boldsymbol{v}_2)$ and, moreover, if $\beta < \Lambda$ by Proposition 3.3, then $\boldsymbol{v}_2$ is a strict local minimum of Φ constrained on $\mathcal{N}$. As a consequence, we have the mountain pass geometry between $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}_2$ on $\mathcal{N}$. We define the set of all continuous paths joining $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}_2$ on the Nehari manifold by

$$\Gamma = \{\gamma : [0,1] \to \mathcal{N} \text{ continuous} : \gamma(0) = \widetilde{\boldsymbol{u}}, \gamma(1) = \boldsymbol{v}_2\}.$$

Thanks to the mountain pass theorem of Ambrosetti and Rabinowitz (see [9]) there exists a Palais–Smale sequence $\{\boldsymbol{u}_k\} \subset \mathcal{N}$, i.e.,

$$\Phi(\boldsymbol{u}_k) \to c, \quad \nabla_{\mathcal{N}}\Phi(\boldsymbol{u}_k) \to 0,$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\Phi(\gamma(t)). \tag{4.22}$$

Obviously, by (3.3), the sequence $\{\boldsymbol{u}_k\}$ is bounded on $\mathbb{H}$ and we obtain a weakly convergent subsequence $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}^* \in \mathcal{N}$.

The difficulty of the lack of compactness, due to results in the one-dimensional case (see Remark 3.1 (ii)), can be circumvented in a similar way as in the proof of Theorem 4.1, so we omit the full details for brevity. Thus, we find that the weak limit $\boldsymbol{u}^* = (u^*, v^*)$ is an even bound state of (1.3) and, clearly, $\Phi(\boldsymbol{u}^*) > \Phi(\boldsymbol{v}_2)$.

It remains to prove that $\boldsymbol{u}^* > \mathbf{0}$. To do so, let us introduce the problem

$$\begin{cases} -u'' + \lambda_1 u = (u^+)^3 + \beta u^+ v, \\ -v'' + \lambda_2 v = \dfrac{1}{2}v^2 + \dfrac{1}{2}\beta(u^+)^2. \end{cases} \tag{4.23}$$

By the maximum principle, every nontrivial solution $\boldsymbol{u} = (u, v)$ of (4.23) has first component $u \geq 0$ and second component $v > 0$. Let us define its energy functional by

$$\Phi^+(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2 - G_\beta(u^+, v)$$

and consider the corresponding Nehari manifold

$$\mathcal{N}^+ = \{\boldsymbol{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : (\nabla\Phi^+(\boldsymbol{u})|\boldsymbol{u}) = 0\}.$$

Also, we denote

$$I_1^+(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{4}\int_{\mathbb{R}} (u^+)^4\, dx.$$

It is not very difficult to show that the properties proved for Φ and $\mathcal{N}$ still hold for Φ^+ and $\mathcal{N}^+$. Unfortunately, Φ^+ is not $\mathcal{C}^2$, thus Proposition 3.3 (i) does not hold directly for Φ^+. To overcome this difficulty, we are going to prove that $\boldsymbol{v}_2$ is a strict local minimum of Φ^+ constrained on $\mathcal{N}^+$ without using the second derivative of the functional. Note that in a similar way as in (3.7), there holds

$$\boldsymbol{h} = (h_1, h_2) \in T_{\boldsymbol{v}_2}\mathcal{N}^+ \quad \text{if and only if} \quad h_2 \in T_{V_2}\mathcal{N}_2. \tag{4.24}$$

Taking $\boldsymbol{h} \in T_{\boldsymbol{v}_2}\mathcal{N}^+$ with $\|\boldsymbol{h}\| = 1$, we consider $\boldsymbol{v}_\varepsilon = (\varepsilon h_1, V_2 + \varepsilon h_2)$. Obviously, there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon \boldsymbol{v}_\varepsilon \in \mathcal{N}^+$. Thus, we want to prove that there exists $\varepsilon_1 > 0$ such that

$$\Phi^+(t_\varepsilon \boldsymbol{v}_\varepsilon) > \Phi^+(\boldsymbol{v}_2) \quad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

It is convenient to distinguish whether $h_1 = 0$ or not. In the former case, for $h_1 = 0$, we have $\boldsymbol{v}_\varepsilon = (0, V_2 + \varepsilon h_2)$. Hence, $t_\varepsilon \boldsymbol{v}_\varepsilon \in \mathcal{N}^+$ if and only if $t_\varepsilon(V_2 + \varepsilon h_2) \in \mathcal{N}_2$. Furthermore,

$$\Phi^+(t_\varepsilon \boldsymbol{v}_\varepsilon) = I_2(t_\varepsilon(V_2 + \varepsilon h_2)) > I_2(V_2) = \Phi(\boldsymbol{v}_2) = \Phi^+(\boldsymbol{v}_2), \tag{4.25}$$

where the previous inequality holds because V_2 is a strict local minimum of I_2 on $\mathcal{N}_2$.

Let us now consider the case $h_1 \neq 0$. There holds

$$\Phi^+(t_\varepsilon \boldsymbol{v}_\varepsilon) = I_2(t_\varepsilon(V_2 + \varepsilon h_2)) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx. \tag{4.26}$$

By (4.25) and (4.26) it follows that

$$\Phi^+(t_\varepsilon \boldsymbol{v}_\varepsilon) > \Phi^+(\boldsymbol{v}_2) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx. \tag{4.27}$$

To finish, it is sufficient to show that

$$\mathcal{J}(t_\varepsilon \boldsymbol{v}_\varepsilon) := I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx > 0 \quad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

Let $\alpha < 1$ be such that $\alpha > \frac{\beta}{\Lambda}$. By (3.8) and $\beta < \Lambda$ there holds

$$\beta \int_{\mathbb{R}} V_2 (h_1^+)^2\, dx < \alpha \|h_1\|_1^2$$

and, for ε_1 smaller than before (if necessary), we have

$$\beta \int_{\mathbb{R}} (V_2 + \varepsilon h_2)(h_1^+)^2 \, dx < \alpha \|h_1\|_1^2 \quad \text{for all } 0 < \varepsilon < \varepsilon_1. \tag{4.28}$$

Using (4.28) and the Sobolev inequality we obtain

$$\mathcal{J}(t_\varepsilon \boldsymbol{v}_\varepsilon) > \frac{1}{2} t_\varepsilon^2 \varepsilon^2 \|h_1\|_1^2 (1 - \alpha - c t_\varepsilon^2 \varepsilon^2) \quad \text{for a constant } c > 0.$$

Now, taking into account that $t_\varepsilon \to 1$ as $\varepsilon \searrow 0$, we infer that there exists a constant $c_0 > 0$ such that

$$\mathcal{J}(t_\varepsilon \boldsymbol{v}_\varepsilon) > \varepsilon^2 c_0 \|h_1\|_1^2. \tag{4.29}$$

Finally, by (4.27) and (4.29) it follows that

$$\Phi^+(t_\varepsilon \boldsymbol{v}_\varepsilon) > \varepsilon^2 c_0 \|h_1\|_1^2 + \Phi^+(\boldsymbol{v}_2) > \Phi^+(\boldsymbol{v}_2),$$

which proves that $\boldsymbol{v}_2$ is a strict local minimum for Φ^+ on $\mathcal{N}^+$.

From the preceding arguments, it follows that Φ^+ has a mountain pass critical point $\boldsymbol{u}^* \in \mathcal{N}^+$, which gives rise to a solution of (4.23). In particular, one finds that $u, v \geq 0$. In addition, since $\boldsymbol{u}^*$ is a mountain pass critical point, one has that $\Phi(\boldsymbol{u}^*) = \Phi^+(\boldsymbol{u}^*) > \Phi^+(\boldsymbol{v}_2) = \Phi(\boldsymbol{v}_2) > 0$, which implies that $u^* \geq 0$ with $u^* \not\equiv 0$, and by the maximum principle applied to each single equation, we get $u^*, v^* > 0$, hence, $\boldsymbol{u}^* > \boldsymbol{0}$. □

In view of Theorem 4.3 and Theorem 4.5, some remarks are in order.

Remark 4.6. (i) Following the proof of Theorem 4.5, a natural question is what happens in the limit case $\beta = \Lambda$. In that case, $\boldsymbol{u}^*$ could coincide with $\boldsymbol{v}_2$ which is nonnegative, but not positive. Indeed, this is our conjecture in view of the second equation in (6.1), see also Figure 1.

(ii) In the hypotheses of Theorem 4.3 and Theorem 4.5, we have found the coexistence of two positive solutions, the ground state $\tilde{\boldsymbol{u}}$ in Theorem 4.3 and the bound state $\boldsymbol{u}^*$ in Theorem 4.5, proving a nonuniqueness result of positive solutions to (1.3). This is a great difference compared to the more studied system of coupled NLS equations

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, \end{cases}$$

(see, for instance, [4–6, 10, 13, 16, 22, 25, 26, 29, 33, 34, 36–38] and the references therein) for which it is known that there is uniqueness of positive solutions under appropriate conditions on the parameters, including the case $\beta > 0$ small; see, more specifically, [25, 38]. Indeed, for $\beta > 0$ small, the ground state is not positive and it is given by one of the two semitrivial solutions $(U^{(1)}, 0)$ or $(0, U^{(2)})$ depending on whether $\Phi(U^{(1)}, 0)$ is lesser or greater than $\Phi(0, U^{(2)})$, which obviously corresponds to $\lambda_1^{2-\frac{n}{2}} \mu_2 < \lambda_2^{2-\frac{n}{2}} \mu_1$ or $\lambda_1^{2-\frac{n}{2}} \mu_2 > \lambda_2^{2-\frac{n}{2}} \mu_1$, respectively. Here, $U^{(j)}$ is the unique[1] positive radial solution of $-\Delta u_j + \lambda_j u_j = \mu_j u_j^3$ in $W^{1,2}(\mathbb{R}^n)$ for $n = 1, 2, 3$ and $j = 1, 2$.

5 An extended NLS–KdV system with general power nonlinearities

In this section, we want to show that if one considers a more general system than (1.1), (1.3) with more general power nonlinearities, like, for example,

$$\begin{cases} i f_t + f_{xx} + \tau_1 |f|^{q-1} f + \beta f g = 0, \\ g_t + g_{xxx} + \tau_2 |g|^{p-1} g_x + \frac{1}{2} \beta (|f|^2)_x = 0, \end{cases} \tag{5.1}$$

1 See [14, 28] for this uniqueness result.

where τ_1, τ_2, β are real constants, then one can prove the same results of the previous section under appropriate hypotheses. Looking for solutions of (5.1) in the form of (1.2), we find that, for $\mu_1 = \tau_1$, $\mu_2 = \frac{\tau_2}{p}$, the real functions u, v solve the system

$$\begin{cases} -u'' + \lambda_1 u = \mu_1 |u|^{q-1} u + \beta u v, \\ -v'' + \lambda_2 v = \mu_2 |v|^{p-1} v + \dfrac{1}{2}\beta u^2, \end{cases} \tag{5.2}$$

where we consider $\lambda_j, \mu_j > 0$, $j = 1, 2$, $p, q \geq 2$. We take $\beta > 0$ in order to obtain positive solutions, although some results on the existence of bound states also hold true without positivity.

Some of the results in Section 4 hold with minor changes. Notice that, since we look for positive solutions of (5.2), one could consider the term $(|g|^p)_x$ (as in the previous sections, where $p = 2$) instead of $|g|^{p-1} g_x$ in (5.1) and, hence, one would have $|g|^p$ instead of $|g|^{p-1} g$ in (5.2), obtaining the same existence of positive bound and ground states that we will prove in Theorem 5.2 and Theorem 5.3. More general systems than (5.2) will be analyzed in a forthcoming paper.

Note that (5.2) has a unique nonnegative semitrivial solution defined by $\boldsymbol{v}_p = (0, V_p)$, where V_p is the unique positive solution of $-v'' + \lambda_2 v = \mu_2 |v|^{p-1} v$ in H and has the explicit expression

$$V_p(x) = \left[\frac{(p+1)\lambda_2}{2\mu_2 \cosh^2(\frac{p-1}{2}\sqrt{\lambda_2} x)} \right]^{\frac{1}{p-1}}. \tag{5.3}$$

Using similar notation as for (1.3), we denote the associated energy functional of (5.2) by

$$\Phi(\boldsymbol{u}) = J_1(u) + J_2(v) - \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v \, dx, \quad \boldsymbol{u} \in \mathbb{E}, \tag{5.4}$$

where, for $u, v \in E$, we have

$$J_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{\mu_1}{q+1} \int_{\mathbb{R}} |u|^{q+1} \, dx, \quad J_2(v) = \frac{1}{2}\|v\|_2^2 - \frac{\mu_2}{p+1} \int_{\mathbb{R}} |v|^{p+1} \, dx.$$

Also, for $\Psi(\boldsymbol{u}) = (\nabla \Phi(\boldsymbol{u}) | \boldsymbol{u})$, we define the corresponding Nehari manifold as

$$\mathcal{N} = \{\boldsymbol{u} \in \mathbb{H} \setminus \{\boldsymbol{0}\} : \Psi(\boldsymbol{u}) = 0\}.$$

Obviously, for all $\boldsymbol{u} \in \mathcal{N}$, we have

$$(\nabla \Psi(\boldsymbol{u}) | \boldsymbol{u}) = -\|\boldsymbol{u}\|^2 - \mu_1 (q-2) \int_{\mathbb{R}} |u|^{q+1} \, dx - \mu_2 (p-2) \int_{\mathbb{R}} |v|^{p+1} \, dx,$$

thus, $\mathcal{N}$ is a smooth manifold locally near any point $\boldsymbol{u} \neq \boldsymbol{0}$ with $\Psi(\boldsymbol{u}) = 0$. Moreover, $\Phi''(\boldsymbol{0}) = I_1''(0) + I_2''(0)$ is positive definite, so we infer that $\boldsymbol{0}$ is a strict minimum of Φ. As a consequence, $\boldsymbol{0}$ is an isolated point of the set $\{\Psi(\boldsymbol{u}) = 0\}$, proving, on one hand, that $\mathcal{N}$ is a smooth complete manifold of codimension 1 and, on the other hand, that there exists a constant $\rho > 0$ such that

$$\|\boldsymbol{u}\|^2 > \rho \quad \text{for all } \boldsymbol{u} \in \mathcal{N}. \tag{5.5}$$

Then, as for (1.3), where $q = 3$, $p = 2$, one has that $\boldsymbol{u} \in \mathbb{H} \setminus \{\boldsymbol{0}\}$ is a critical point of Φ if and only if $\boldsymbol{u} \in \mathcal{N}$ is a critical point of Φ constrained on $\mathcal{N}$. Furthermore, for all $\boldsymbol{u} \in \mathcal{N}$, we have

$$\Phi(\boldsymbol{u}) = \frac{1}{6}\|\boldsymbol{u}\|^2 + \left(\frac{1}{3} - \frac{1}{q+1}\right)\mu_1 \int_{\mathbb{R}} |u|^{q+1} \, dx + \left(\frac{1}{3} - \frac{1}{p+1}\right)\mu_2 \int_{\mathbb{R}} |v|^{p+1} \, dx$$

and, clearly, by (5.5) and the previous identity, Φ on $\mathcal{N}$ is bounded below for every $2 \leq p < \infty$, $2 \leq q < \infty$.

Proposition 5.1. *For Λ defined by* (3.8),

(i) *if $\beta \leq \Lambda$, then $\boldsymbol{v}_p$ is a strict local minimum of Φ constrained on $\mathcal{N}$;*

(ii) *for any $\beta > \Lambda$, $\boldsymbol{v}_p$ is a saddle point of Φ constrained on $\mathcal{N}$. Moreover,* $\inf_{\mathcal{N}} \Phi < \Phi(\boldsymbol{v}_2)$.

The proof is a straightforward calculation of the proof of Proposition 3.3. Furthermore, defining U_q as the unique positive solution of $-u'' + \lambda_1 u = \mu_1 |u|^{q-1} u$ in H (given by (5.3) substituting p by q), we have the following theorem.

Theorem 5.2. *Assume that* $2 \le p < \infty$, $2 \le q < \infty$. *Then,*
(i) *if* $\beta > \Lambda$, *then system* (5.2) *has a positive even ground state* $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$;
(ii) *there exists* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$ *and* $\beta = \varepsilon\tilde{\beta} > 0$, *system* (5.2) *has an even bound state* $\boldsymbol{u}_\varepsilon > \mathbf{0}$ *with* $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0 = (U_q, V_p)$ *as* $\varepsilon \to 0$.

Proof. We can adapt, with appropriate modifications, the ideas in the proof of Theorem 4.1 since the nonlinearity $|v|^{p+1}$ is even, while in Theorem 4.1 the nonlinearity on v is v^3 (odd). This proves part (i).

Part (ii) follows by a small modification of the ideas of Theorem 4.4, by the same reasoning as above. □

Concerning the existence of ground states for any $\beta > 0$, one can follow the proof of Theorem 4.3, which also holds true with a restriction on the power exponent $q > 2p - 2$ and which trivially holds for (1.3), where $q = 3$, $p = 2$. Thus, using this property, it is not difficult to also show the existence of positive bound states for $0 < \beta < \Lambda$ as in Theorem 4.5. We enunciate these results in the following theorem.

Theorem 5.3. *Assume that* $2 \le p < \infty$, $2 \le q < \infty$ *and, moreover,* $q > 2p - 2$. *Then,*
(i) *there exists* $M > 0$ *such that if* $\lambda_2 > M$, *then system* (5.2) *has an even ground state* $\tilde{\mathbf{u}} > \mathbf{0}$ *for every* $\beta > 0$,
(ii) *if* $\lambda_2 > M$ *and* $0 < \beta < \Lambda$, *then there exists an even bound state* $\boldsymbol{u}^* > \mathbf{0}$ *with* $\Phi(\boldsymbol{u}^*) > \Phi(\boldsymbol{v}_2)$.

Remark 5.4. (i) The restriction $q > 2p - 2$ appears when one tries to prove that

$$\Phi(t(V_p, V_p)) < \Phi(\boldsymbol{v}_p)$$

for $t(V_p, V_p) \in \mathcal{N}$. It does not seem to be optimal. Another test function different from $t(V_p, V_p)$ could circumvent this difficulty.
(ii) When $p = 2$, $\mu_2 = p + 1$, Dias, Figueira and Oliveira in [20, Theorem 4.1] impose $\beta > 3$ to obtain even bound states. In our Theorem 5.2 (i), following the idea of Remark 3.4, it is easy to see that it holds for $\beta > 3 - a$ for some constant $a > 0$ when $\lambda_2 > \lambda_1$, obtaining positive even bound and ground states.
(iii) Note that in Theorem 5.2 and Theorem 5.3 we have $2 \le p < \infty$, $2 \le q < \infty$, obtaining positive even bound and ground states, in contrast with [20, Theorem 4.1] where $2 < q < 5$, $p \in \{2, 3, 4\}$, $\mu_2 = p + 1$ and with [3, Theorem 1.1] where $2 \le q < 5$, $2 \le p < 5$ with p a rational number with odd denominator, where the authors obtained nonnegative even bound states in the former and positive even bound states in the latter case.

6 Further results

In this last section, we show some results for explicit solutions and we point out some remarks and open problems. As a conclusion, we study some extended systems with three or more equations.

6.1 Explicit solutions

In the particular case $0 < \beta < \frac{1}{6}$, $\lambda_2 = 4\lambda_1 + \frac{1}{12}\beta(1 - 6\beta)$, there exists a nontrivial explicit solution[2] $\boldsymbol{u}_\beta = (u_\beta, v_\beta)$ of (1.3) defined by

$$u_\beta(x) = \frac{\sqrt{2\lambda_1(1 - 6\beta)}}{\cosh(\sqrt{\lambda_1}x)}, \quad v_\beta(x) = \frac{12\lambda_1}{\cosh^2(\sqrt{\lambda_1}x)}.$$

2 Although the results in this subsection can be established for the more general system (5.2), we restrict ourselves to (1.3) for brevity.

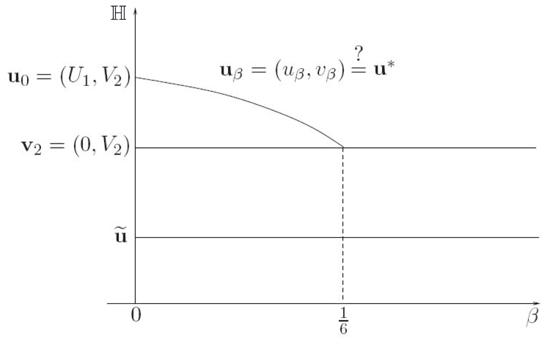

Figure 1. Suggestive bifurcation diagram of the family $\boldsymbol{u}_\beta = (u_\beta, v_\beta)$.

Clearly, one has

$$\lim_{\beta \searrow 0} \boldsymbol{u}_\beta = \boldsymbol{u}_0 = (U_1, V_2), \quad \lim_{\beta \nearrow \frac{1}{6}} \boldsymbol{u}_\beta = \boldsymbol{v}_2 = (0, V_2), \tag{6.1}$$

where U_1, V_2 are defined by (4.12), (3.6), respectively. Then, the family $\{\boldsymbol{u}_\beta : 0 < \beta < \frac{1}{6}\}$ joins $\boldsymbol{u}_0 = (U_1, V_2)$ with $\boldsymbol{v}_2$.

Remark 6.1. (i) If we would have $\Phi(\boldsymbol{v}_2) \geq \Phi(\boldsymbol{u}_\beta)$ in the range $0 < \beta < \min\{\frac{1}{6}, \Lambda\}$, then we would be able to prove the existence of a positive even bound state $\boldsymbol{u}^*$ with $\Phi(\boldsymbol{u}^*) > \max\{\Phi(\boldsymbol{u}_\beta), \Phi(\boldsymbol{v}_2)\} = \Phi(\boldsymbol{v}_2)$ and, in particular, we would have a result on the nonuniqueness of positive solutions by a different way compared to the previous sections. Unfortunately, if $0 < \beta < \frac{1}{6}$, then

$$\begin{aligned} \Phi(\boldsymbol{v}_2) &= \frac{1}{6}\|\boldsymbol{v}_2\|^2 = \frac{1}{12}\int_{\mathbb{R}} V_2^3\,dx = \frac{9}{2}\lambda_2^3 \int_{\mathbb{R}} \frac{1}{\cosh^6\big(\frac{\sqrt{\lambda_2}}{2}x\big)}\,dx \\ &= \frac{24}{5}\lambda_2^{\frac{5}{2}} = \frac{24}{5}\Big[4\lambda_1 + \frac{1}{12}\beta(1-6\beta)\Big]^{\frac{5}{2}} \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \Phi(\boldsymbol{u}_\beta) &= \frac{1}{6}\|\boldsymbol{u}_\beta\|^2 + \frac{1}{12}\int_{\mathbb{R}} u_\beta^4\,dx \\ &= \frac{1}{6}(\|u_\beta\|_1^2 + \|v_\beta\|_2^2) + \frac{1}{12}\int_{\mathbb{R}} u_\beta^4\,dx \\ &= \frac{1}{6}\int_{\mathbb{R}} u_\beta^4\,dx + \frac{\beta}{6}\int_{\mathbb{R}} u_\beta^2 v_\beta\,dx + \frac{1}{12}\int_{\mathbb{R}} v_\beta^3\,dx + \frac{\beta}{12}\int_{\mathbb{R}} u_\beta^2 v_\beta\,dx + \frac{1}{12}\int_{\mathbb{R}} u_\beta^4\,dx \\ &= \frac{1}{4}\int_{\mathbb{R}} u_\beta^4\,dx + \frac{\beta}{4}\int_{\mathbb{R}} u_\beta^2 v_\beta\,dx + \frac{1}{12}\int_{\mathbb{R}} v_\beta^3\,dx \\ &= \lambda_1^2(1-6\beta)\int_{\mathbb{R}} \frac{1}{\cosh^4(\sqrt{\lambda_1}x)}\,dx + 144\lambda_1^3 \int_{\mathbb{R}} \frac{1}{\cosh^6(\sqrt{\lambda_1}x)}\,dx \\ &= \frac{4}{3}\lambda_1^{\frac{3}{2}}(1-6\beta) + \frac{768}{5}\lambda_1^{\frac{5}{2}}. \end{aligned} \tag{6.3}$$

Comparing both energies, it is not difficult to show that $\Phi(\boldsymbol{v}_2) < \Phi(\boldsymbol{u}_\beta)$ for every $0 < \beta < \frac{1}{6}$.

(ii) In the setting in which the explicit solution $\boldsymbol{u}_\beta$ of (1.3) exists, i.e., $0 < \beta < \frac{1}{6}$, $\lambda_2 = 4\lambda_1 + \frac{1}{12}\beta(1-6\beta)$, under the hypotheses of Theorem 4.3 and Theorem 4.5, we have the multiplicity of positive solutions, i.e., the ground state $\tilde{\boldsymbol{u}}$ and the bound state $\boldsymbol{u}^*$. We conjecture that $\boldsymbol{u}_\beta$ coincides with $\boldsymbol{u}^*$ and, hence, the suggestive bifurcation diagram in Figure 1 holds not only for $\boldsymbol{u}_\beta$, but also for $\boldsymbol{u}^*$.

6.2 Some systems with more than two equations

In this last subsection, we deal with some extended systems of (1.3) to more than two equations[3], but we also consider (1.3) in other dimensions.

Note that system (1.1) has no sense for dimensions $n = 2, 3$. However, it makes sense to extend system (1.3) to other dimensions. Moreover, for dimensions $n = 2, 3$, the results of the previous sections can be established with minor changes for the system

$$\begin{cases} -\Delta u + \lambda_1 u = u^3 + \beta u v, \\ -\Delta v + \lambda_2 v = \frac{1}{2} v^2 + \frac{1}{2}\beta u^2, \end{cases} \tag{6.4}$$

by working on the corresponding Sobolev space $E = W^{1,2}(\mathbb{R}^n)$, $n = 2, 3$, and its radial subspace $H = E_r$. In particular, Theorem 4.1, Theorem 4.3, Theorem 4.4 and Theorem 4.5 hold, yielding the corresponding positive bound and ground state solutions which are radially symmetric in this case.

Remark 6.2. (i) For $n = 2, 3$, there is no lack of compactness since we have the compact embedding of the radial Sobolev Space H for all $2 < s < 2^*$ (see [30]), where $2^* = \infty$ if $n = 2$ and $2^* = \frac{2n}{n-2}$ for $n = 3$, which allows us to prove the Palais–Smale condition[4] working on $\mathbb{H}$.

(ii) Following some ideas by Ambrosetti and Colorado in [6], as Liu and Zheng cited in [32], they proved a partial result on the existence of solutions to the corresponding system (1.3) for dimensions $n = 2, 3$. More precisely, in [32], the authors showed that the infimum of the energy functional on the corresponding Nehari manifold (defined on the radial Sobolev space) is achieved by a nonnegative bound state, although it was not shown that the infimum on the Nehari manifold is a ground state, i.e., the least energy solution of the functional that we have proved here for $n = 1, 2, 3$. Also, in [32], the existence of other bound states was not investigated, which is done in this paper, not only in the noncritical dimensions $n = 2, 3$, but also in the one-dimensional case $n = 1$, which is the relevant case for applications in physics dealing with the interaction between the short and long capillary-gravity water waves.

System (6.4) can be seen as the stationary system of two coupled NLS–NLS equations when one looks for solitary wave solutions and (u, v) are the corresponding standing wave solutions. It is well known that systems of NLS–NLS time-dependent equations have applications in some aspects of optics and in the Hartree–Fock theory for Bose–Einstein condensates, among other physical phenomena; see, for instance, the earlier works [1, 4–7, 10, 23, 29, 34, 36, 37] and also the more recent ones [13, 26, 33, 38] (the list is far from complete) and the references therein. See also [12, 35] for some recent results dealing with NLS equations and [24] for other related results, including higher-order NLS equations.

By the above discussion, one can motivate, from the application point of view, the study of the system of NLS–KdV–KdV equations

$$\begin{cases} -\Delta u + \lambda_0 u = u^3 + \beta_1 u v_1 + \beta_2 u v_2, \\ -\Delta v_1 + \lambda_1 v_1 = \frac{1}{2} v_1^2 + \frac{1}{2}\beta_1 u^2, \\ -\Delta v_2 + \lambda_2 v_2 = \frac{1}{2} v_2^2 + \frac{1}{2}\beta_2 u^2. \end{cases} \tag{6.5}$$

This system can also be seen as a perturbation of (6.4) if $n = 2, 3$ or as a perturbation of (1.3) if $n = 1$, when $|\beta_1|$ or $|\beta_2|$ is small.

3 Similar extensions of (5.2) to other dimensions, as (6.4) extends (1.3), can be considered (at least in the subcritical framework with $p, q < 2^*$ defined in Remark 6.2), proving results similar to Theorem 6.3 and Theorem 6.5.

4 In a similar way as in [6, Lemma 3.2].

In the following, we use the same notation as in the previous sections, for example, $\mathbb{H} = H \times H \times H$, $\mathbb{E} = E \times E \times E$, $\mathbf{0} = (0, 0, 0)$,

$$\Phi(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2 - \frac{1}{4}\int_{\mathbb{R}} u^4\,dx - \frac{1}{6}\int_{\mathbb{R}} (v_1^3 + v_2^3)\,dx - \frac{1}{2}\int_{\mathbb{R}} u^2(\beta_1 v_1 + \beta_2 v_2)\,dx, \tag{6.6}$$

$$\mathcal{N} = \{\boldsymbol{u} \in \mathbb{E} \setminus \{\mathbf{0}\} : (\Phi'(\boldsymbol{u})|\boldsymbol{u}) = 0\} \tag{6.7}$$

and so on.

Let U^*, V_j^* be the unique positive radial solutions of $-\Delta u + \lambda u = u^3$, $-\Delta v + \lambda_j v = \frac{1}{2}v^2$ in E, $j = 1, 2$, respectively, see [14, 28]. Then, we have the following theorem.

Theorem 6.3. *There exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and $\beta_j = \varepsilon\widetilde{\beta}_j > 0$, $j = 1, 2$, system* (6.5) *has a radial bound state $\boldsymbol{u}_\varepsilon^*$ with $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}_0^* = (U^*, V_1^*, V_2^*)$ as $\varepsilon \to 0$. Moreover, if $\beta_j > 0$ for $j = 1, 2$, then $\boldsymbol{u}_\varepsilon^* > \mathbf{0}$.*

The proof is similar to the proof of Theorem 4.4 with appropriate modifications, so we omit it for brevity.

We can also prove the existence of a positive and radial ground state of (6.5) when the coupling parameters β_j, $j = 1, 2$, are sufficiently large. To do so, we define

$$\Lambda_j = \inf_{\varphi \in H\setminus\{0\}} \frac{\|\varphi\|_0^2}{\int_{\mathbb{R}^n} V_j^* \varphi^2\,dx}, \quad j = 1, 2, \tag{6.8}$$

where $\|\cdot\|_0$ is the norm in E with λ_0.

Remark 6.4. The unique nonnegative semitrivial solutions of (6.5) are given by $\boldsymbol{v}_1^* = (0, V_1^*, 0)$, $\boldsymbol{v}_2^* = (0, 0, V_2^*)$ and $\boldsymbol{v}_{12}^* = (0, V_1^*, V_2^*)$ with V_j^*, $j = 1, 2$, defined before Theorem 6.3.

Concerning the ground states of (6.5), the first result is the following theorem.

Theorem 6.5. *If $\beta_j > \Lambda_j$ for $j = 1, 2$, then* (6.5) *has a positive radial ground state $\widetilde{\boldsymbol{u}}$.*

Proof. As in Proposition 3.3, using now that $\beta_j > \Lambda_j$, $j = 1, 2$, one can show that both $\boldsymbol{v}_1^*$, $\boldsymbol{v}_2^*$ are saddle points of the energy functional Φ (defined by (6.6)) constrained on the Nehari manifold $\mathcal{N}$ (defined by (6.6)). Then,

$$c = \inf_{\mathcal{N}} \Phi < \min\{\Phi(\boldsymbol{v}_1^*), \Phi(\boldsymbol{v}_2^*)\} < \Phi(\boldsymbol{v}_{12}^*) = \Phi(\boldsymbol{v}_1^*) + \Phi(\boldsymbol{v}_2^*). \tag{6.9}$$

By Ekeland's variational principle, there exists a Palais–Smale sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\boldsymbol{u}_k) \to c, \quad \nabla_{\mathcal{N}}\Phi(\boldsymbol{u}_k) \to 0. \tag{6.10}$$

If $n = 2, 3$, then $\{\boldsymbol{u}_k\}$ satisfies the Palais–Smale condition, see Remark 6.2 (i). Thus, there exists a convergent subsequence, denoted again by $\{\boldsymbol{u}_k\}$, such that $\boldsymbol{u}_k \to \widetilde{\boldsymbol{u}}$. Arguing in a similar way as in the proof of Theorem 4.1, we have that $\widetilde{\boldsymbol{u}} \geq \mathbf{0}$ and, moreover, by the properties of the Schwarz symmetrization, one can prove that

$$c = \Phi(\widetilde{\boldsymbol{u}}) = \min\{\Phi(\boldsymbol{u}) : \boldsymbol{u} \in \mathbb{E}, \Phi'(\boldsymbol{u}) = 0\}.$$

To prove the positivity of $\widetilde{\boldsymbol{u}}$, if one supposes that for the first component holds $\widetilde{u} \equiv 0$, since the only nonnegative solutions of (6.5) are the semitrivial solutions defined in Remark 6.4, we obtain a contradiction to (6.9). Furthermore, if the second or third component vanishes, then $\widetilde{\boldsymbol{u}}$ must be $\mathbf{0}$ and this is not possible since $\Phi|_{\mathcal{N}}$ is bounded from below by a positive constant like as in (3.4). Then, $\mathbf{0}$ is an isolated point of the set $\{\boldsymbol{u} \in \mathbb{H} : (\Phi'(\boldsymbol{u})|\boldsymbol{u}) = 0\}$, proving that $\mathcal{N}$ is a complete manifold. Finally, the maximum principle gives that $\widetilde{\boldsymbol{u}} > \mathbf{0}$.

In the one-dimensional case, the lack of compactness discussed in Remark 3.1 (ii) can be circumvented as in the proof of Theorem 4.1, proving the result. □

Furthermore, one can show similar results to Theorem 4.3 and Theorem 4.5 in the setting of (6.5). To do so, we first prove an auxiliary result in the following proposition.

Proposition 6.6. *Assume that $\beta_j < \Lambda_j$, $j = 1, 2$, and, moreover, $\frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2} < 1$. Then, $\boldsymbol{v}^*_{12}$ is a strict local minimum of Φ constrained on $\mathcal{N}$.*

Proof. Notice that since $\boldsymbol{v}^*_{12}$ is a critical point of Φ, we have

$$D^2\Phi_{\mathcal{N}}(\boldsymbol{v}^*_{12})[\boldsymbol{h}]^2 = \Phi''(\boldsymbol{v}^*_{12})[\boldsymbol{h}]^2 \quad \text{for all } \boldsymbol{h} \in T_{\boldsymbol{v}^*_{12}}\mathcal{N}.$$

We denote

$$I_j(v) = \frac{1}{2}\|v\|^2_{\lambda_j} - \frac{1}{6}\int_{\mathbb{R}^n} v^3\,dx.$$

Then, using that V^*_j is a strict local minimum of I_j, we have that there exist two positive constants c_1, c_2 such that, for $\boldsymbol{h} = (h_0, h_1, h_2) \in T_{\boldsymbol{v}^*_{12}}\mathcal{N}$, we have

$$\Phi''(\boldsymbol{v}^*_{12})[\boldsymbol{h}]^2 \geq c_1\|h_1\|^2_1 + c_2\|h_2\|^2_2 + \|h_0\|^2 - \beta_1\int_{\mathbb{R}^n} h_0^2 V^*_1\,dx - \beta_2\int_{\mathbb{R}^n} h_0^2 V^*_2\,dx. \tag{6.11}$$

From (6.11) and $\dfrac{\beta_1}{\Lambda_1} + \dfrac{\beta_2}{\Lambda_2} < 1$ we infer that

$$\Phi''(\boldsymbol{v}^*_{12})[\boldsymbol{h}]^2 \geq c_1\|h_1\|^2_1 + c_2\|h_2\|^2_2 + \left(1 - \frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2}\right)\|h_0\|^2,$$

which proves the result. ☐

Theorem 6.7. *Assume that $\beta_1, \beta_2 > 0$ for λ_1, λ_2 large enough. Then,*
(i) *there exists a radial ground state $\tilde{\boldsymbol{u}} > \boldsymbol{0}$;*
(ii) *if additionally $\frac{\beta_1}{\Lambda_1} + \frac{\beta_2}{\Lambda_2} < 1$, then there exists a radial bound state $\boldsymbol{u}^* > 0$. Furthermore, $\Phi(\boldsymbol{u}^*) > \Phi(\boldsymbol{v}^*_{12}) = \Phi(\boldsymbol{v}^*_1) + \Phi(\boldsymbol{v}^*_2)$.*

Proof. The proof of (i) follows in a similar way to the one of Theorem 4.3 with appropriate changes, so we omit the details for brevity. With respect to (ii), using Proposition 6.6 one can appropriately modify the arguments of the proof of Theorem 4.5 to obtain the result, so once again we omit the details. ☐

Remark 6.8. It is easy to extend these results to systems with any number $N > 3$ of equations as in the system

$$\begin{cases} -\Delta u + \lambda_0 u = u^3 + \displaystyle\sum_{k=1}^{N-1}\beta_k u v_k, \\ -\Delta v_j + \lambda_j v_j = \dfrac{1}{2}v_j^2 + \dfrac{1}{2}\beta_j u^2, \quad j = 1, \dots, N-1. \end{cases} \tag{6.12}$$

For example, there exists a positive radial ground state of (6.12) provided

$$\beta_k > \Lambda_k = \inf_{\varphi\in H\setminus\{0\}} \frac{\|\varphi\|^2_0}{\int_{\mathbb{R}} V^*_k\varphi^2\,dx}, \quad k = 1, \dots, N-1,$$

where V^*_k is the unique positive radial solution of $\Delta v + \lambda_k v = \frac{1}{2}v^2$ in E, $k = 1, \dots, N-1$.

A natural extension of (1.3) to more than two equations, different from (6.5) and (6.12), is the system of NLS–NLS–KdV equations

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = u_1^3 + \beta_{12}u_1u_2^2 + \beta_{13}u_1 v, \\ -\Delta u_2 + \lambda_2 u_2 = u_2^3 + \dfrac{1}{2}\beta_{12}u_1^2u_2 + \beta_{23}u_2 v, \\ -\Delta v + \lambda v = \dfrac{1}{2}v^2 + \dfrac{1}{2}\beta_{13}u_1^2 + \dfrac{1}{2}\beta_{23}u_2^2. \end{cases} \tag{6.13}$$

Here, we obtain the following bifurcation result for this system in a similar way as in Theorem 4.4 and Theorem 6.3.

Theorem 6.9. *There exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ and $\beta_{jk} = \varepsilon\tilde{\beta}_{jk}$, $k = 1, 2$, $j = 2, 3$, $k \neq j$, system (6.13) has a radial bound state $\boldsymbol{u}^*_\varepsilon$ with $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*_0 = (U_1, U_2, V)$ as $\varepsilon \to 0$. Moreover, if all $\beta_{kj} > 0$, then $\boldsymbol{u}^*_\varepsilon > \boldsymbol{0}$.*

Here, U_j is the unique positive radial solution of $-\Delta u + \lambda_j u = u^3$ in E, $j = 1, 2$, and V is the corresponding positive radial solution to $-\Delta v + \lambda v = \frac{1}{2}v^2$ in E.

Note that the nonnegative radial semitrivial solution $(0, 0, V)$ is a strict local minimum of the associated energy functional constrained on the corresponding Nehari manifold provided

$$\beta_{j3} < \Lambda_j = \inf_{\varphi \in H\setminus\{0\}} \frac{\|\varphi\|^2_{\lambda_j}}{\int_{\mathbb{R}^n} V\varphi^2\, dx}, \qquad j = 1, 2,$$

while, if either $\beta_{13} > \Lambda_1$ or $\beta_{23} > \Lambda_2$, then $(0, 0, V)$ is a saddle critical point of Φ on $\mathcal{N}$.

There also exist semitrivial solutions coming from the solutions studied in Section 4, with the first or the second component identically equal to 0. This fact makes the analysis of (6.13) different with respect to the previous studied systems (6.5) and (6.12).

Finally, one could study more general extended systems of the type of (6.5) and (6.13) with $N = m + \ell$, for example, m-NLS and ℓ-KdV coupled equations with $m, \ell \geq 2$ in the one-dimensional case or N-NLS equations if $n = 2, 3$. A careful analysis of this kind of systems, including (6.13), will be carried out in a forthcoming paper.

Funding: The research of the author was partially supported by the Ministry of Economy and Competitiveness of Spain and FEDER under research project MTM2013-44123-P.

References

[1] N. Akhmediev and A. Ankiewicz, *Solitons: Nonlinear Pulses and Beams*, Opt. Quantum Electron. 5, Chapman & Hall, London, 1997.
[2] J. Albert and J. Angulo Pava, Existence and stability of ground-state solutions of a Schrödinger–KdV system, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), no. 5, 987–1029.
[3] J. Albert and S. Bhattarai, Existence and stability of a two-parameter family of solitary waves for an NLS–KdV system, *Adv. Differential Equations* **18** (2013), no. 11–12, 1129–1164.
[4] A. Ambrosetti, A note on nonlinear Schrödinger systems: Existence of a-symmetric solutions, *Adv. Nonlinear Stud.* **6** (2006), no. 2, 149–155.
[5] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 7, 453–458.
[6] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, *J. Lond. Math. Soc. (2)* **75** (2007), no. 1, 67–82.
[7] A. Ambrosetti, E. Colorado and D. Ruiz, Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations, *Calc. Var. Partial Differential Equations* **30** (2007), no. 1, 85–112.
[8] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^n$, Progr. Math. 240, Birkhäuser, Basel, 2005.
[9] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
[10] T. Bartsch and Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, *J. Partial Differ. Equ.* **19** (2006), no. 3, 200–207.
[11] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Ration. Mech. Anal.* **82** (1983), no. 4, 313–345.
[12] A. Bahrouni, H. Ounaies and V. D. Rădulescu, Infinitely many solutions for a class of sublinear Schrödinger equations with indefinite potentials, *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), no. 3, 445–465.
[13] Z. Chen and W. Zou, An optimal constant for the existence of least energy solutions of a coupled Schrödinger system, *Calc. Var. Partial Differential Equations* **48** (2013), no. 3–4, 695–711.
[14] C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Ration. Mech. Anal.* **46** (1972), 81–95.
[15] E. Colorado, Existence results for some systems of coupled fractional nonlinear Schrödinger equations, in: *Recent Trends in Nonlinear Partial Differential Equations. II. Stationary Problems* (Perugia 2012), Contemp. Math. 595, American Mathematical Society, Providence (2013), 135–150.
[16] E. Colorado, Positive solutions to some systems of coupled nonlinear Schrödinger equations, *Nonlinear Anal.* **110** (2014), 104–112.

[17] E. Colorado, Existence of bound and ground states for a system of coupled nonlinear Schrödinger–KdV equations, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 6, 511–516.
[18] A. J. Corcho and F. Linares, Well-posedness for the Schrödinger–Korteweg–de Vries system, *Trans. Amer. Math. Soc.* **359** (2007), no. 9, 4089–4106.
[19] J.-P. Dias, M. Figueira and F. Oliveira, Existence of bound states for the coupled Schrödinger–KdV system with cubic nonlinearity, *C. R. Math. Acad. Sci. Paris* **348** (2010), no. 19–20, 1079–1082.
[20] J.-P. Dias, M. Figueira and F. Oliveira, Well-posedness and existence of bound states for a coupled Schrödinger–gKdV system, *Nonlinear Anal.* **73** (2010), no. 8, 2686–2698.
[21] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353.
[22] D. G. de Figueiredo and O. Lopes, Solitary waves for some nonlinear Schrödinger systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), no. 1, 149–161.
[23] M. Funakoshi and M. Oikawa, The resonant interaction between a long internal gravity wave and a surface gravity wave packet, *J. Phys. Soc. Japan* **52** (1983), no. 1, 1982–1995.
[24] V. A. Galaktionov, E. L. Mitidieri and S. I. Pohozaev, *Blow-up for higher-order parabolic, hyperbolic, dispersion and Schrödinger equations*, Monogr. Res. Notes Math., CRC Press, Boca Raton, 2015.
[25] N. Ikoma, Uniqueness of positive solutions for a nonlinear elliptic system, *NoDEA Nonlinear Differential Equations Appl.* **16** (2009), no. 5, 555–567.
[26] N. Ikoma and K. Tanaka, A local mountain pass type result for a system of nonlinear Schrödinger equations, *Calc. Var. Partial Differential Equations* **40** (2011), no. 3–4, 449–480.
[27] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math. 1150, Springer, Berlin, 1985.
[28] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.* **105** (1989), no. 3, 243–266.
[29] T.-C. Lin and J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Comm. Math. Phys.* **255** (2005), no. 3, 629–653; erratum, *Comm. Math. Phys.* **277** (2008), no. 2, 573–576.
[30] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, *J. Funct. Anal.* **49** (1982), no. 3, 315–334.
[31] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), no. 4, 223–283.
[32] C. Liu and Y. Zheng, On soliton solutions to a class of Schrödinger–KdV systems, *Proc. Amer. Math. Soc.* **141** (2013), no. 10, 3477–3484.
[33] Z. Liu and Z.-Q. Wang, Ground states and bound states of a nonlinear Schrödinger system, *Adv. Nonlinear Stud.* **10** (2010), no. 1, 175–193.
[34] L. A. Maia, E. Montefusco and B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, *J. Differential Equations* **229** (2006), no. 2, 743–767.
[35] D. Repovš, Stationary waves of Schrödinger-type equations with variable exponent, *Anal. Appl. (Singap.)* **13** (2015), no. 6, 645–661.
[36] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Comm. Math. Phys.* **271** (2007), no. 1, 199–221.
[37] J. Wei and T. Weth, Nonradial symmetric bound states for a system of coupled Schrödinger equations, *Atti Accad. Naz. Lincei Rend. Lincei (9) Mat. Appl.* **18** (2007), no. 3, 279–293.
[38] J. Wei and W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, *Commun. Pure Appl. Anal.* **11** (2012), no. 3, 1003–1011.